

QUOTIENTS OF CLUSTER CATEGORIES

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ABSTRACT. Higher cluster categories were recently introduced as a generalization of cluster categories.

This paper shows that in Dynkin types A and D , half of all higher cluster categories are actually just quotients of cluster categories. The other half can be obtained as quotients of 2-cluster categories, the “lowest” type of higher cluster categories.

Hence, in Dynkin types A and D , all higher cluster phenomena are implicit in cluster categories and 2-cluster categories. In contrast, the same is not true in Dynkin type E .

0. INTRODUCTION

This paper is about the connection between quotient categories and cluster categories, so let me start by explaining these two notions.

Quotient categories come in a number of different flavours. The one to be considered here is probably the most basic: Let \mathbf{A} be an additive category with a class of objects \mathbb{Y} . For objects A and B of \mathbf{A} , denote by $\mathbb{Y}(A, B)$ all the morphisms from A to B which factor through an object of \mathbb{Y} . Then the quotient category $\mathbf{A}_{\mathbb{Y}}$ has the same objects as \mathbf{A} , and its morphism spaces are defined by

$$\mathbf{A}_{\mathbb{Y}}(A, B) = \mathbf{A}(A, B) / \mathbb{Y}(A, B).$$

Cluster categories and the more general u -cluster categories, parametrized by the positive integer u , were introduced in [12], [19], [31], [39], and [40]. They underlie a representation theoretical viewpoint on the theory of cluster algebras introduced and developed in [10], [22], [23], and [24]. Cluster categories have generated a strong activity in recent years, not least because of their connection to finite dimensional algebras and tilting theory, see [1], [2], [3], [5], [6], [12], [13], [14], [15],

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[16], [17], [18], [19], [20], [21], [25], [26], [28], [29], [30], [31], [32], [33], [34], [38], [39], and [40]. One particular aspect of this connection is the importance of quivers in both subjects, and in particular of Dynkin quivers of types A , D , and E .

The present paper shows that quotient categories permit a bridge between cluster categories and u -cluster categories. Specifically, it will be proved that in Dynkin types A and D , half of all u -cluster categories are actually just quotients of cluster categories, in a sense of the word “half” which will be made precise.

In the language of u -cluster categories, a cluster category is the same thing as a 1-cluster category, so an equivalent way to formulate the statement is to say that, in types A and D , half of all u -cluster categories are quotients of 1-cluster categories. This will be complemented by a proof that the other half of the u -cluster categories are quotients of 2-cluster categories. These statements are shown in Corollaries 4.3 and 4.5. In contrast, the corresponding result in type E is not true, see Remark 4.7.

The results will be proved in the following strong sense: It will be established that the quotient categories in question are triangulated categories which are equivalent *as triangulated categories* to the relevant u -cluster categories.

As a backdrop to this, I will show more generally, under some technical assumptions, that if T is a triangulated category with a class of objects \mathbb{X} , then the quotient category $\mathsf{T}_{\mathbb{X}}$ is triangulated if and only if \mathbb{X} is equal to its image under the Auslander-Reiten translation of T (Theorem 2.3). When this is the case, I will prove that as a translation quiver, the Auslander-Reiten quiver of $\mathsf{T}_{\mathbb{X}}$ can be obtained from the Auslander-Reiten quiver of T by deleting the vertices corresponding to objects in \mathbb{X} (Theorem 3.2). These results permit the deletion of orbits of the Auslander-Reiten translation from the Auslander-Reiten quiver of a triangulated category, without destroying the property of being triangulated.

Note that, while it is classical that Verdier quotients of triangulated categories are triangulated, it seems to be less well known that the present, simpler type of quotient categories sometimes have a triangulated structure. Some references do exist, see [7, sec. 7] and [30, sec. 4], but no systematic exploration seems to have taken place.

The paper is organized as follows: Sections 1 and 2 develop the theory of triangulated quotients of triangulated categories. Section 3 considers

the Auslander-Reiten theory of triangulated quotients. And Section 4 applies the theory to u -cluster categories.

Background material. Let me round off the introduction with some facts about Krull-Schmidt categories and their quotients.

Let k be an algebraically closed field and let \mathbf{A} be a k -linear category which is Krull-Schmidt, that is, each object of \mathbf{A} is a sum of indecomposable objects which are unique up to isomorphism.

The radical rad of \mathbf{A} determines the subquotient

$$\text{Irr}(M, N) = \text{rad}(M, N) / \text{rad}^2(M, N),$$

see [4, p. 178 and p. 228]. The Auslander-Reiten (AR) quiver of \mathbf{A} has one vertex for each isomorphism class of indecomposable objects and $\dim_k \text{Irr}(M, N)$ arrows from the vertex of M to the vertex of N .

Now let \mathbb{Y} be a class of objects of \mathbf{A} closed under isomorphisms, direct sums, and direct summands. Then the quotient category $\mathbf{A}_{\mathbb{Y}}$ makes sense, and the useful content of the following lemma is folklore.

- Lemma 0.1.** (i) *If $A \cong B$ in $\mathbf{A}_{\mathbb{Y}}$, then there exist objects X, Y in \mathbb{Y} such that in \mathbf{A} , the object A is isomorphic to a direct summand of $B \oplus Y$ and the object B is isomorphic to a direct summand of $A \oplus X$.*
- (ii) *The category $\mathbf{A}_{\mathbb{Y}}$ is Krull-Schmidt and the isomorphism classes of indecomposable objects in $\mathbf{A}_{\mathbb{Y}}$ correspond to the isomorphism classes of indecomposable objects in \mathbf{A} which are not in \mathbb{Y} .*
- (iii) *The AR quiver of $\mathbf{A}_{\mathbb{Y}}$ is obtained from the AR quiver of \mathbf{A} by deleting the vertices corresponding to objects of \mathbb{Y} along with the arrows into or out of such vertices.*

(i) and (ii) are straightforward, with (ii) following from (i).

(iii) is obtained by combining (ii) and the following: If M and N are indecomposable objects of \mathbf{A} which are not in \mathbb{Y} , then $\text{rad}_{\mathbf{A}_{\mathbb{Y}}}^n(M, N) \cong \text{rad}_{\mathbf{A}}^n(M, N) / \mathbb{Y}(M, N)$ whence

$$\text{Irr}_{\mathbf{A}_{\mathbb{Y}}}(M, N) \cong \text{Irr}_{\mathbf{A}}(M, N).$$

1. PRETRIANGULATED QUOTIENT CATEGORIES

As a stepping stone towards triangulated quotient categories, this section shows how to equip quotient categories with a pretriangulated structure.

Recall from [9, sec. II.1] the notion of a pretriangulated category. This is an additive category equipped with some data: There is an endofunctor σ and a class of diagrams of the form $A \rightarrow B \rightarrow C \rightarrow \sigma A$ called distinguished right-triangles. There is also an endofunctor ω and a class of diagrams of the form $\omega Z \rightarrow X \rightarrow Y \rightarrow Z$ called distinguished left-triangles. The distinguished right-triangles satisfy the axioms of a triangulated category, except that σ is not required to be an equivalence of categories. Similarly for the distinguished left-triangles. Finally, there are some compatibility conditions, most importantly that (σ, ω) is an adjoint pair of functors.

Setup 1.1. Let T be a triangulated category with suspension functor Σ . Let \mathbb{X} be a class of objects of T , closed under isomorphisms, which is both preenveloping and precovering.

Recall that for \mathbb{X} to be preenveloping means that each object M has an \mathbb{X} -preenvelope, that is a morphism $M \rightarrow X_M$ with X_M in \mathbb{X} such that each morphism $M \rightarrow X$ with X in \mathbb{X} factors through $M \rightarrow X_M$,

$$\begin{array}{ccc} M & \longrightarrow & X_M \\ & \searrow & \vdots \\ & & X. \end{array} \quad (1)$$

Dually, for \mathbb{X} to be precovering means that each object M has an \mathbb{X} -precover $X^M \rightarrow M$,

$$\begin{array}{ccc} X & & \\ \vdots \searrow & \searrow & \\ X^M & \longrightarrow & M. \end{array}$$

Under Setup 1.1, the quotient category $\mathsf{T}_{\mathbb{X}}$ can be turned into a pretriangulated category as follows:

First, to get the endofunctors σ and ω , pick, for each M in T , an \mathbb{X} -preenvelope $M \rightarrow X_M$ and an \mathbb{X} -precover $X^M \rightarrow M$. Complete to distinguished triangles in T ,

$$M \rightarrow X_M \rightarrow \sigma M \rightarrow \Sigma M \quad (2)$$

and

$$\Sigma^{-1} M \rightarrow \omega M \rightarrow X^M \rightarrow M. \quad (3)$$

This defines objects σM and ωM . To turn σ and ω into functors, note that if $M \xrightarrow{\mu} N$ is a morphism in T , then there is a commutative

diagram

$$\begin{array}{ccccccc}
 M & \longrightarrow & X_M & \longrightarrow & \sigma M & \longrightarrow & \Sigma M \\
 \mu \downarrow & & \xi \downarrow & & \downarrow s & & \downarrow \Sigma\mu \\
 N & \longrightarrow & X_N & \longrightarrow & \sigma N & \longrightarrow & \Sigma N,
 \end{array} \tag{4}$$

where ξ exists because X_N is in \mathbb{X} and $M \rightarrow X_M$ is an \mathbb{X} -preenvelope, and s exists by one of the axioms for the triangulated category \mathcal{T} . Now denote by $\underline{\mu}$ and \underline{s} the morphisms in $\mathcal{T}_{\mathbb{X}}$ corresponding to μ and s and set $\sigma(\underline{\mu}) = \underline{s}$. This turns σ into an endofunctor of $\mathcal{T}_{\mathbb{X}}$, and the dual method works for ω .

Secondly, to get distinguished right- and left-triangles, let $M \xrightarrow{\mu} N$ be an \mathbb{X} -monomorphism in \mathcal{T} , that is, a morphism such that each morphism $M \rightarrow X$ with X in \mathbb{X} factors through $M \xrightarrow{\mu} N$,

$$\begin{array}{ccc}
 M & \xrightarrow{\mu} & N \\
 & \searrow & \vdots \\
 & & X.
 \end{array} \tag{5}$$

Then μ can be extended to a distinguished triangle $M \xrightarrow{\mu} N \xrightarrow{\nu} P \rightarrow \Sigma M$ which fits into a commutative diagram

$$\begin{array}{ccccccc}
 M & \xrightarrow{\mu} & N & \xrightarrow{\nu} & P & \longrightarrow & \Sigma M \\
 \parallel & & n \downarrow & & \downarrow \pi & & \parallel \\
 M & \longrightarrow & X_M & \longrightarrow & \sigma M & \longrightarrow & \Sigma M,
 \end{array} \tag{6}$$

where n exists because X_M is in \mathbb{X} and $M \xrightarrow{\mu} N$ is an \mathbb{X} -monomorphism, and π exists by one of the axioms for the triangulated category \mathcal{T} . Declare

$$M \xrightarrow{\mu} N \xrightarrow{\nu} P \xrightarrow{\pi} \sigma M$$

to be a distinguished right-triangle in $\mathcal{T}_{\mathbb{X}}$.

Dually, let $M \rightarrow N$ be an \mathbb{X} -epimorphism,

$$\begin{array}{ccc}
 X & & \\
 \vdots \downarrow & \searrow & \\
 M & \longrightarrow & N.
 \end{array} \tag{7}$$

Then $M \rightarrow N$ can be extended to a distinguished triangle, and proceeding dually to the above construction gives the distinguished left-triangles in $\mathcal{T}_{\mathbb{X}}$.

With this data, the following theorem holds.

Theorem 1.2. *The quotient category $\mathsf{T}_{\mathbb{X}}$ is pretriangulated.*

Proof. Morally speaking, this is [7, thm. 7.2], which, however, is stated with different assumptions. The proof can be carried through in the manner of the proof of [8, thm. 3.1].

It is helpful to start by establishing that the following construction is an alternative way of getting the distinguished right-triangles in $\mathsf{T}_{\mathbb{X}}$: Given a morphism $M \xrightarrow{\mu} N$ in T , consider the distinguished triangle $M \rightarrow X_M \rightarrow \sigma M \rightarrow \Sigma M$ from equation (2). The octahedral axiom for T gives a way to embed it into a commutative diagram where the rows are distinguished triangles,

$$\begin{array}{ccccccc} M & \longrightarrow & X_M & \longrightarrow & \sigma M & \longrightarrow & \Sigma M \\ \mu \downarrow & & \downarrow & & \parallel & & \downarrow \Sigma\mu \\ N & \xrightarrow{\nu} & P & \xrightarrow{\pi} & \sigma M & \longrightarrow & \Sigma N. \end{array}$$

Up to isomorphism, the distinguished right-triangles in $\mathsf{T}_{\mathbb{X}}$ are now precisely the diagrams which can be obtained as

$$M \xrightarrow{\mu} N \xrightarrow{\nu} P \xrightarrow[-\pi]{} \sigma M.$$

Note the sign change on the last arrow. □

Remark 1.3. Standard arguments show that none of the choices involved in constructing the pretriangulated structure on $\mathsf{T}_{\mathbb{X}}$ make any difference.

That is, choosing the distinguished triangles (2) and (3) or the morphisms in the diagrams (4) and (6) differently gives an equivalent structure of pretriangulated category.

2. TRIANGULATED QUOTIENT CATEGORIES

This section considers the pretriangulated quotient category $\mathsf{T}_{\mathbb{X}}$ from Section 1 and shows, under some technical assumptions, that it is triangulated if and only if \mathbb{X} is equal to its image under the Auslander-Reiten (AR) translation of T .

Setup 2.1. Let k be an algebraically closed field and let T be a k -linear triangulated category with suspension functor Σ and Serre functor S , which has finite dimensional Hom spaces and split idempotents. Let \mathbb{X} be a class of objects of T , closed under isomorphisms, direct sums, and direct summands, which is both precovering and preenveloping.

The conditions imply that T is Krull-Schmidt, see [37, p. 52]. Recall that the Serre functor S is an autoequivalence of T for which there are natural isomorphisms

$$\mathsf{T}(A, B) \cong \mathsf{T}(B, SA)^\vee,$$

where $(-)^\vee = \mathrm{Hom}_k(-, k)$. The Serre functor S determines the AR translation $\tau = \Sigma^{-1} \circ S$.

For the following lemma, note that I will write “ $\tau\mathbb{X} = \mathbb{X}$ ” as a shorthand for “the set of objects isomorphic to objects in $\tau\mathbb{X}$ is equal to \mathbb{X} ”. The notions of \mathbb{X} -monomorphism and \mathbb{X} -epimorphism are described by diagrams (5) and (7).

Lemma 2.2. *Suppose $\tau\mathbb{X} = \mathbb{X}$. If $A \rightarrow B \rightarrow C \rightarrow$ is a distinguished triangle in T , then $A \rightarrow B$ is an \mathbb{X} -monomorphism if and only if $B \rightarrow C$ is an \mathbb{X} -epimorphism.*

Proof. Let X run through \mathbb{X} . For each X , the distinguished triangle induces a long exact sequence

$$\mathsf{T}(X, B) \rightarrow \mathsf{T}(X, C) \rightarrow \mathsf{T}(X, \Sigma A) \rightarrow \mathsf{T}(X, \Sigma B).$$

For $B \rightarrow C$ to be an \mathbb{X} -epimorphism is the same as for the first arrow in the long exact sequence always to be surjective. This is the same as for the second arrow always to be zero, which is again the same as for the third arrow always to be injective.

Using Serre duality, the third arrow can be identified with

$$\mathsf{T}(A, \Sigma^{-1}SX)^\vee \rightarrow \mathsf{T}(B, \Sigma^{-1}SX)^\vee$$

which is injective if and only if

$$\mathsf{T}(B, \Sigma^{-1}SX) \rightarrow \mathsf{T}(A, \Sigma^{-1}SX)$$

is surjective. But this last arrow is

$$\mathsf{T}(B, \tau X) \rightarrow \mathsf{T}(A, \tau X).$$

For this always to be surjective is the same as for $A \rightarrow B$ to be a $(\tau\mathbb{X})$ -monomorphism, that is, an \mathbb{X} -monomorphism. \square

The pretriangulated structure of $\mathsf{T}_{\mathbb{X}}$ is a triangulated structure if and only if the functor σ is an autoequivalence.

Theorem 2.3. *The pretriangulated structure of $\mathsf{T}_{\mathbb{X}}$ is a triangulated structure if and only if $\tau\mathbb{X} = \mathbb{X}$.*

Proof. First assume that $\tau\mathbb{X} = \mathbb{X}$; I must show that σ is an autoequivalence.

It follows directly from the definitions (diagrams (1) and (5)) that an \mathbb{X} -preenvelope is simply an \mathbb{X} -monomorphism $M \rightarrow X$ with X in \mathbb{X} ; similarly, an \mathbb{X} -precover is an \mathbb{X} -epimorphism $X \rightarrow M$ with X in \mathbb{X} . Consider the distinguished triangle (2),

$$M \rightarrow X_M \rightarrow \sigma M \rightarrow \Sigma M. \quad (8)$$

The morphism $M \rightarrow X_M$ is an \mathbb{X} -preenvelope and hence an \mathbb{X} -monomorphism, and so by Lemma 2.2 the morphism $X_M \rightarrow \sigma M$ is an \mathbb{X} -epimorphism and hence an \mathbb{X} -precover. Now, $\omega\sigma M$ is computed by completing such a precover to a distinguished triangle as in equation (3), but completing $X_M \rightarrow \sigma M$ to a distinguished triangle just recovers (8), so $\omega\sigma M \cong M$ in $\mathbf{T}_{\mathbb{X}}$.

This isomorphism is easily shown to be natural. Similarly, there is a natural isomorphism $\sigma\omega M \cong M$, and so σ is an autoequivalence whence $\mathbf{T}_{\mathbb{X}}$ is triangulated.

Next assume that $\mathbf{T}_{\mathbb{X}}$ is triangulated, that is, σ (and hence also ω) is an autoequivalence; I must show $\tau\mathbb{X} = \mathbb{X}$. This amounts to seeing $\tau\mathbb{X} \subseteq \mathbb{X}$ and $\tau^{-1}\mathbb{X} \subseteq \mathbb{X}$, and I will give the proof of the first of these since the second one is dual.

Let X be an indecomposable object in \mathbb{X} and consider the AR triangle

$$\tau X \xrightarrow{t} Y \rightarrow X \rightarrow \quad (9)$$

in \mathbf{T} . If τX is in \mathbb{X} then I am done, so let me suppose not. When τX is not in \mathbb{X} , a morphism in \mathbf{T} from τX to an object in \mathbb{X} cannot be a split monomorphism, so any such morphism factors through t which is hence an \mathbb{X} -monomorphism. This means that there is a distinguished right-triangle

$$\tau X \xrightarrow{t} Y \rightarrow X \rightarrow \sigma\tau X$$

in $\mathbf{T}_{\mathbb{X}}$, and this is isomorphic to

$$\tau X \xrightarrow{t} Y \rightarrow 0 \rightarrow \sigma\tau X.$$

Since $\mathbf{T}_{\mathbb{X}}$ is triangulated, this shows that \underline{t} is an isomorphism, and Lemma 0.1(i) gives that Y is a direct summand of $\tau X \coprod X'$ in \mathbf{T} for some X' in \mathbb{X} . Hence Y has the form $\tau X \coprod X_1$ for some X_1 in \mathbb{X} . Note that τX is forced to be among the indecomposable direct summands of Y , since Y would otherwise be zero in $\mathbf{T}_{\mathbb{X}}$ forcing τX to be zero in $\mathbf{T}_{\mathbb{X}}$ and thereby contradicting that τX is not in \mathbb{X} .

But now t has the form

$$\tau X \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} \tau X \coprod X_1,$$

and since X_1 is in \mathbb{X} and hence zero in $\mathsf{T}_{\mathbb{X}}$, the morphism \underline{t} equals \underline{u} . However, \underline{t} and hence \underline{u} is an isomorphism, and so an invertible element of the ring $\text{End}_{\mathsf{T}_{\mathbb{X}}}(\tau X)$. This ring is a quotient of $\text{End}_{\mathsf{T}}(\tau X)$, and \underline{u} is the image of u . But τX is indecomposable so $\text{End}_{\mathsf{T}}(\tau X)$ is local, and it follows that since \underline{u} is invertible in the quotient, u must itself be invertible.

This implies that t is split, contradicting that (9) is an AR triangle. \square

3. THE AUSLANDER-REITEN THEORY OF QUOTIENT CATEGORIES

This section continues to work under Setup 2.1.

The quotient category $\mathsf{T}_{\mathbb{X}}$ is Krull-Schmidt by Lemma 0.1(ii), and if $\tau\mathbb{X} = \mathbb{X}$ then $\mathsf{T}_{\mathbb{X}}$ is triangulated by Theorem 2.3. This section shows that when $\mathsf{T}_{\mathbb{X}}$ is triangulated then it has AR triangles and its AR quiver can be computed, as a translation quiver, from the AR quiver of T by deleting the vertices corresponding to objects of \mathbb{X} along with the arrows into or out of such vertices. This sharpens Lemma 0.1(iii).

Proposition 3.1. *Suppose that the AR translation τ of T satisfies $\tau\mathbb{X} = \mathbb{X}$.*

Then the triangulated category $\mathsf{T}_{\mathbb{X}}$ has a Serre functor \underline{S} , and the AR translation $\sigma^{-1} \circ \underline{S}$ of $\mathsf{T}_{\mathbb{X}}$ is equal to the functor $\underline{\tau}$ on $\mathsf{T}_{\mathbb{X}}$ which is induced by the functor τ on T .

Proof. Since $\mathsf{T}_{\mathbb{X}}$ is triangulated, ω is equivalent to σ^{-1} and the distinguished triangle (3) from Section 1 reads

$$\Sigma^{-1}M \rightarrow \sigma^{-1}M \rightarrow X^M \rightarrow M. \quad (10)$$

Let N be an object of $\mathsf{T}_{\mathbb{X}}$ and note that N can also be viewed as an object of T . Since $X^M \rightarrow M$ is an \mathbb{X} -precover, it is easy to see that the cokernel of $\mathsf{T}(N, X^M) \rightarrow \mathsf{T}(N, M)$ is $\mathsf{T}_{\mathbb{X}}(N, M)$, and so there is an exact sequence

$$\mathsf{T}(N, X^M) \rightarrow \mathsf{T}(N, M) \rightarrow \mathsf{T}_{\mathbb{X}}(N, M) \rightarrow 0. \quad (11)$$

For each L in T , the distinguished triangle (10) gives a long exact sequence

$$\mathsf{T}(X^M, \Sigma^{-1}L) \rightarrow \mathsf{T}(\sigma^{-1}M, \Sigma^{-1}L) \rightarrow \mathsf{T}(M, L) \rightarrow \mathsf{T}(X^M, L). \quad (12)$$

Since $X^M \rightarrow M$ is an \mathbb{X} -epimorphism, Lemma 2.2 gives that $\sigma^{-1}M \rightarrow X^M$ is an \mathbb{X} -monomorphism and hence an \mathbb{X} -preenvelope, and it is again easy to see that the cokernel of

$$\mathsf{T}(X^M, \Sigma^{-1}L) \rightarrow \mathsf{T}(\sigma^{-1}M, \Sigma^{-1}L)$$

is $\mathsf{T}_{\mathbb{X}}(\sigma^{-1}M, \Sigma^{-1}L)$. However, by (12) this cokernel is isomorphic to the kernel of

$$\mathsf{T}(M, L) \rightarrow \mathsf{T}(X^M, L),$$

so there is an exact sequence

$$0 \rightarrow \mathsf{T}_{\mathbb{X}}(\sigma^{-1}M, \Sigma^{-1}L) \rightarrow \mathsf{T}(M, L) \rightarrow \mathsf{T}(X^M, L).$$

Setting $L = SN$ and taking the k -linear dual gives an exact sequence

$$\mathsf{T}(X^M, SN)^{\vee} \rightarrow \mathsf{T}(M, SN)^{\vee} \rightarrow \mathsf{T}_{\mathbb{X}}(\sigma^{-1}M, \Sigma^{-1}(SN))^{\vee} \rightarrow 0. \quad (13)$$

But the sequences (11) and (13) fit together in a commutative diagram

$$\begin{array}{ccccccc} \mathsf{T}(N, X^M) & \longrightarrow & \mathsf{T}(N, M) & \longrightarrow & \mathsf{T}_{\mathbb{X}}(N, M) & \longrightarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ \mathsf{T}(X^M, SN)^{\vee} & \longrightarrow & \mathsf{T}(M, SN)^{\vee} & \longrightarrow & \mathsf{T}_{\mathbb{X}}(\sigma^{-1}M, \Sigma^{-1}(SN))^{\vee} & \longrightarrow & 0, \end{array}$$

where the two first isomorphisms are by the definition of the Serre functor S and the third isomorphism follows from the first two. So there is a natural isomorphism

$$\mathsf{T}_{\mathbb{X}}(N, M) \cong \mathsf{T}_{\mathbb{X}}(M, \sigma(\Sigma^{-1}SN))^{\vee}.$$

Hence $\mathsf{T}_{\mathbb{X}}$ has a right Serre functor \underline{S} which is given on objects by $\underline{S}N = \sigma(\Sigma^{-1}SN)$. Similar computations show that the same formula gives a left Serre functor, and hence a Serre functor, see [36, sec. I.1].

The formula implies that the AR translation $\sigma^{-1} \circ \underline{S}$ of $\mathsf{T}_{\mathbb{X}}$ is induced by $\Sigma^{-1}S = \tau$. \square

Let Γ be the AR quiver of T . For the following theorem, recall that the category T is called standard if it is equivalent to $\text{add } k(\Gamma)$, the additive closure of the mesh category $k(\Gamma)$, see [11, def. 5.1].

Theorem 3.2. *Suppose $\tau\mathbb{X} = \mathbb{X}$.*

- (i) *The triangulated category $\mathsf{T}_{\mathbb{X}}$ has AR triangles.*

- (ii) *As a translation quiver, the AR quiver Γ' of $\mathsf{T}_{\mathbb{X}}$ is obtained from the AR quiver Γ of T by deleting the vertices corresponding to objects of \mathbb{X} along with the arrows into or out of such vertices.*
- (iii) *If T is standard, then so is $\mathsf{T}_{\mathbb{X}}$.*

Proof. (i) By Proposition 3.1, the category $\mathsf{T}_{\mathbb{X}}$ has a Serre functor, and hence also AR triangles by [36, prop. I.2.3].

(ii) Lemma 0.1(iii) implies that the vertices and arrows of Γ' can be obtained from those of Γ as described. The AR translation of Γ descends to the AR translation of Γ' by Proposition 3.1.

(iii) Suppose that T is standard. This means that there is an equivalence of categories $\mathsf{T} \simeq \text{add } k(\Gamma)$. Let \mathbb{V} be the set of objects in $\text{add } k(\Gamma)$ which corresponds to \mathbb{X} , and let Γ' be the AR quiver of $\mathsf{T}_{\mathbb{X}}$. There is an equivalence $\mathsf{T}_{\mathbb{X}} \simeq (\text{add } k(\Gamma))_{\mathbb{V}}$, so it is enough to see that there is an equivalence $(\text{add } k(\Gamma))_{\mathbb{V}} \simeq \text{add } k(\Gamma')$.

However, there is a functor $\text{add } k(\Gamma) \rightarrow \text{add } k(\Gamma')$ given by sending each object of \mathbb{V} to 0. This functor factors through a functor $(\text{add } k(\Gamma))_{\mathbb{V}} \rightarrow \text{add } k(\Gamma')$ which is the required equivalence. \square

4. CLUSTER CATEGORIES

This section applies the methods of the previous sections to show in Corollaries 4.3 and 4.5 that in Dynkin types A and D , all u -cluster categories are triangulated quotients of 1- and 2-cluster categories.

The following result is due to Amiot [1, thm. 1.1.1 and thm. 7.0.5 plus proof].

Theorem 4.1. *Let U and V be k -linear triangulated categories with finite dimensional Hom spaces and split idempotents. Suppose that U and V are of algebraic origin, standard, connected, and have only a finite number of isomorphism classes of indecomposable objects.*

Then U and V have AR triangles, and if the AR quivers of U and V are isomorphic as translation quivers, then U and V are equivalent as triangulated categories.

Note that in particular, U or V could be taken to be a u -cluster category of finite type. Such categories are of algebraic origin by the theory of [31, sec. 9.3] (see also [1, sec. 7.3]), and they are standard by [1, prop. 6.1.1].

4.a. Type A.

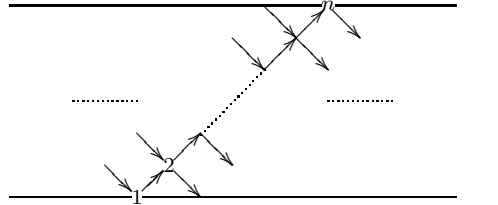
Theorem 4.2. *Let $u \geq v$ be positive integers for which $u \equiv v \pmod{2}$. Let m, n be positive integers such that*

$$u(m+1) = v(n+1).$$

Then the u -cluster category of type A_m is triangulated equivalent to a quotient category of the v -cluster category of type A_n .

Proof. The proof will work by appealing to the theory of triangulated quotient categories developed above, and to Theorem 4.1. The categories which will be inserted into Theorem 4.1 will be (1) the u -cluster category of type A_m , and (2) a suitably constructed quotient of the v -cluster category of type A_n . As remarked above, the conditions of Theorem 4.1 hold for u -cluster categories of finite type; I will show along the way that they also hold for the quotient category in question.

The AR quiver of $D(kA_n)$ is $\mathbb{Z}A_n$ by [27, cor. 4.5]; let me visualize this as a horizontal band n vertices wide,



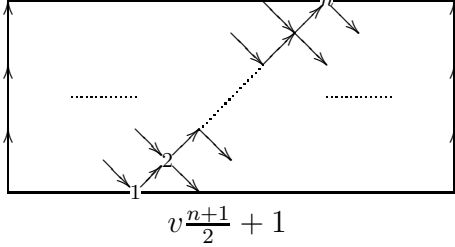
Declare the distance between two horizontally neighbouring vertices to be one unit. The action of τ^{-1} on the AR quiver is to shift one unit to the right. The action of Σ is to shift $\frac{n+1}{2}$ units to the right, and reflect in the horizontal centre line. Both claims follow from [35, table p. 359]. Hence the action of $\tau^{-1}\Sigma^v$ on the AR quiver is to shift $v\frac{n+1}{2} + 1$ units to the right, and reflect in the horizontal centre line if v is odd.

Let T denote the v -cluster category of type A_n , that is,

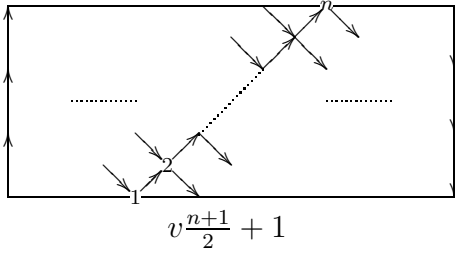
$$\mathsf{T} = D(kA_n)/\tau^{-1}\Sigma^v.$$

The AR quiver of T is the AR quiver of $D(kA_n)$ modulo the action of $\tau^{-1}\Sigma^v$ by [12, prop. 1.3]. Hence the AR quiver of T is $\mathbb{Z}A_n$ modulo an automorphism given by shifting $v\frac{n+1}{2} + 1$ units to the right, and reflecting in the horizontal centre line if v is odd.

This means that if v is even, then the AR quiver of T has the form


(14)

where the vertical ends of the rectangle are identified, with the orientation indicated by the arrows. The number below the quiver indicates side length. And if v is odd, then the AR quiver of T has the form


(15)

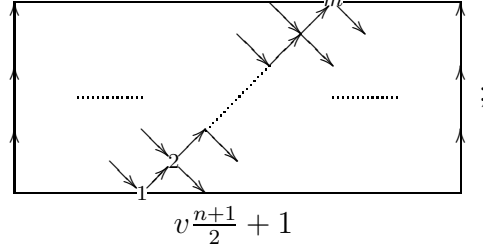
where the vertical ends of the rectangle are again identified, with the orientation indicated by the arrows.

Let me now prove the theorem for u and v even. Here the AR quiver of T is given by diagram (14). Consider the indecomposable objects in a band $n - m$ vertices wide along the bottom of the AR quiver, and let \mathbb{X} denote add of these. Since \mathbb{X} is add of a finite set of objects, it is precovering and preenveloping, and since τ preserves the set of objects in question, it is clear that $\tau\mathbb{X} = \mathbb{X}$.

Let me check the conditions of Theorem 4.1 for the quotient category $\mathsf{T}_{\mathbb{X}}$. Theorem 2.3 says that it is triangulated. It is clear that $\mathsf{T}_{\mathbb{X}}$ is k -linear and has finite dimensional Hom spaces. It is an exercise to check that $\mathsf{T}_{\mathbb{X}}$ has split idempotents. Since T is of algebraic origin, so is $\mathsf{T}_{\mathbb{X}}$. Since T is standard, $\mathsf{T}_{\mathbb{X}}$ is standard by Theorem 3.2(iii).

Finally, by Theorem 3.2(ii), the AR quiver of $\mathsf{T}_{\mathbb{X}}$ is obtained by deleting from the AR quiver of T the band $n - m$ vertices wide along the bottom.

Therefore the AR quiver of $\mathsf{T}_{\mathbb{X}}$ is m vertices wide and has the form



in particular, $\mathsf{T}_{\mathbb{X}}$ is connected and has only a finite number of isomorphism classes of indecomposable objects. This shows that $\mathsf{T}_{\mathbb{X}}$ satisfies the conditions of Theorem 4.1.

Now note that

$$v \frac{n+1}{2} + 1 = u \frac{m+1}{2} + 1$$

because $v(n+1) = u(m+1)$. Since u is even, this implies that the AR quiver of $\mathsf{T}_{\mathbb{X}}$ is precisely the AR quiver of the u -cluster category of type A_m . Hence Theorem 4.1 with U equal to the u -cluster category of type A_m and V equal to $\mathsf{T}_{\mathbb{X}}$ says that these two categories are triangulated equivalent.

In other words, the u -cluster category of type A_m is triangulated equivalent to $\mathsf{T}_{\mathbb{X}}$ which is a quotient category of T , the v -cluster category of type A_n ; this is the desired result.

Next the proof for u and v odd. Here the AR quiver of T is given by diagram (15). The equation $v(n+1) = u(m+1)$ forces the difference $n - m$ to be even. Let \mathbb{X} be add of the indecomposable objects in two bands $\frac{n-m}{2}$ vertices wide along the top and bottom of the AR quiver. Then arguments like the ones above show that the u -cluster category of type A_m is triangulated equivalent to the quotient category $\mathsf{T}_{\mathbb{X}}$, again as desired. \square

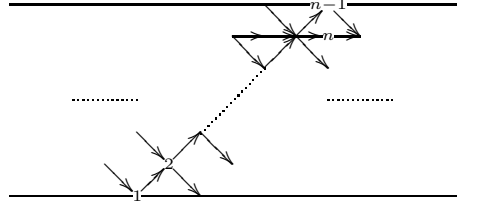
Corollary 4.3. *Each u -cluster category of type A is triangulated equivalent to a quotient of a 1- or a 2-cluster category of type A .*

Proof. This is clear from Theorem 4.2: If u is odd then set $v = 1$ and $n = u(m+1) - 1$, and if u is even then set $v = 2$ and $n = \frac{u}{2}(m+1) - 1$. \square

The corollary shows that in a sense, in type A , every phenomenon of u -cluster categories is already implicit in 1- and 2-cluster categories. Both 1- and 2-cluster categories are needed since the AR quiver of a

1-cluster category is a Möbius band while that of a 2-cluster category is a wreath, whence neither can be obtained from the other.

4.d. **Type D.** The AR quiver of $D(kD_n)$ is $\mathbb{Z}D_n$ by [27, cor. 4.5]; let me visualize this as a horizontal band,



The action of τ^{-1} on the AR quiver is to shift one unit to the right. The action of Σ is to shift $n - 1$ units to the right, and switch the two ‘exceptional’ vertices $n - 1$ and n at the top if n is odd. Both claims again follow from [35, table p. 359]. Hence the action of $\tau^{-1}\Sigma^v$ on the AR quiver is to shift $v(n - 1) + 1$ units to the right, and switch the exceptional vertices if v and n are both odd.

Let T denote the v -cluster category of type D_n , that is,

$$\mathsf{T} = D(kD_n)/\tau^{-1}\Sigma^v.$$

The AR quiver of T is the AR quiver of $D(kD_n)$ modulo the action of $\tau^{-1}\Sigma^v$ by [12, prop. 1.3]. Hence the AR quiver of T is $\mathbb{Z}D_n$ modulo an automorphism given by shifting $v(n - 1) + 1$ units to the right, and switching the exceptional vertices if v and n are both odd.

This means that the AR quiver of T has the form

$$v(n - 1) + 1 \tag{16}$$

where the vertical ends of the rectangle are identified. The number below the quiver indicates side length.

The action of the AR translation on the quiver requires an explanation: If v and n are both odd, then $\tau^{-1}\Sigma^v$ switches the exceptional vertices; otherwise, it does not. This does not make any difference to the way the quiver looks because the exceptional vertices are attached to the rest of the quiver in a symmetrical way. But it does mean that in the

two cases, the action of τ^{-1} on $\mathbb{Z}D_n$ induces different actions of τ^{-1} on the quiver (16). Namely, if v and n are both odd so the exceptional vertices are switched, then τ^{-1} also switches between these vertices at the vertical ends of the rectangle, and so τ^{-1} has *one long* orbit of exceptional vertices. But if v or n is even so the exceptional vertices are not switched, then τ^{-1} does not switch between these vertices at the vertical ends of the rectangle, and so τ^{-1} has *two shorter* orbits of exceptional vertices.

By considering the indecomposable objects in a band $n - m$ vertices wide along the bottom of the AR quiver and following the same strategy as the proof of Theorem 4.2, the following theorem can now be proved.

Theorem 4.4. *Let $u \geq v$ be positive integers and let $m, n \geq 4$ be integers such that*

$$u(m - 1) = v(n - 1).$$

Suppose that if u, m are both odd, then v, n are both odd.

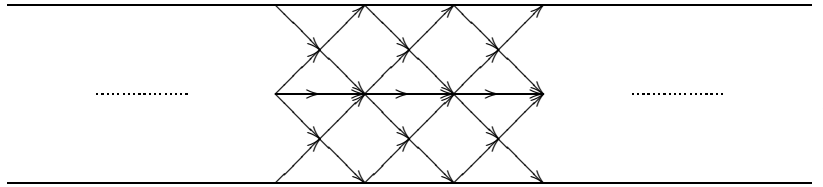
Then the u -cluster category of type D_m is triangulated equivalent to a quotient category of the v -cluster category of type D_n .

Corollary 4.5. *Each u -cluster category of type D is triangulated equivalent to a quotient of a 1- or a 2-cluster category of type D .*

Proof. This follows from Theorem 4.4: If u is odd then set $v = 1$ and $n = u(m - 1) + 1$, and if u is even then set $v = 2$ and $n = \frac{u}{2}(m - 1) + 1$. \square

The corollary shows that in type D , just as in type A , every phenomenon of v -cluster categories is implicit in 1- and 2-cluster categories. Both 1- and 2-cluster categories are again needed because of the different actions of the AR translation on the exceptional vertices.

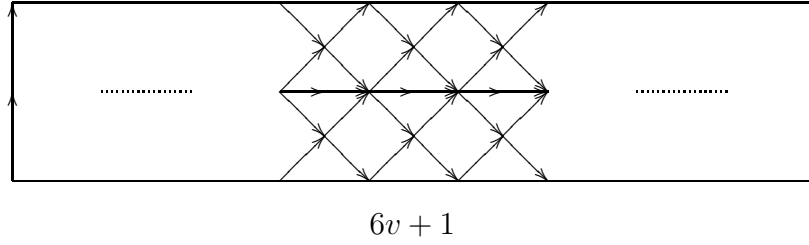
4.e. **Type E.** The AR quiver of $D(kE_n)$ is $\mathbb{Z}E_n$ by [27, cor. 4.5]. Let me use E_6 for illustrative purposes,



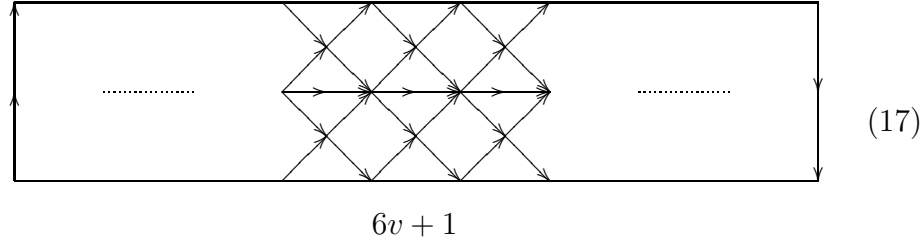
For both E_6 , E_7 , and E_8 , the action of τ^{-1} on the AR quiver is to shift one unit to the right. For E_6 , the action of Σ is to shift 6 units to the right, and reflect in the horizontal centre line. For E_7 , the action of Σ

is to shift 9 units to the right, and for E_8 , the action is to shift 15 units to the right. See [35, table p. 359]. Hence the action of $\tau^{-1}\Sigma^v$ on the AR quiver is the following: For E_6 , it shifts $6v + 1$ units to the right, and reflects in the horizontal centre line if v is odd. For E_7 , it shifts $9v + 1$ units to the right, and for E_8 , it shifts $15v + 1$ units to the right.

Just as in the previous sections, this permits me to compute the AR quiver of $D(kE_n)/\tau^{-1}\Sigma^v$, the v -cluster category of type E_n . For instance, in the case of E_6 , the quiver is



if v is even and



if v is odd. In the cases of E_7 and E_8 , the ends of the rectangle always have the same orientations, and the horizontal lengths are $9v + 1$ and $15v + 1$, respectively.

By deleting one or two rows of vertices from the AR quiver in type E_8 or one row of vertices from the AR quiver in type E_7 , I can get down to types E_6 and E_7 . The resulting theorem is the following.

- Theorem 4.6.**
- (i) Let u and v be positive integers with $3u = 5v$. Then the u -cluster category of type E_7 is triangulated equivalent to a quotient category of the v -cluster category of type E_8 .
 - (ii) Let u and v be positive integers with u even and $2u = 5v$. Then the u -cluster category of type E_6 is triangulated equivalent to a quotient category of the v -cluster category of type E_8 .
 - (iii) Let u and v be positive integers with u even and $2u = 3v$. Then the u -cluster category of type E_6 is triangulated equivalent to a quotient category of the v -cluster category of type E_7 .

In parts (ii) and (iii) of the theorem, u is required to be even to avoid that the AR quiver of the u -cluster category of type E_6 is the Möbius band of figure (17).

Remark 4.7. In contrast to the situation in types A and D , Theorem 4.6 does not provide for arbitrary u -cluster categories of types E_6 and E_7 to be quotients of the 1- and 2-cluster categories of type E_8 .

Indeed, the method used in types A and D was to take a 1- or a 2-cluster category of type A_n or D_n for some large n , then trim its AR quiver to a smaller width. The same idea cannot work in type E , because the AR quivers of v -cluster categories of type E have a fixed, small width.

It appears that in type E , phenomena of general v -cluster categories are not implicit in the 1- and 2-cluster situations.

4.z. Mixed type. By employing different deletions of vertices, it is also possible to go from type D_n to type A_m for $m < n$, and from types E_6 , E_7 , E_8 to types A_2 through A_7 and types D_4 through D_7 . The details of this are left to the reader.

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